

# BOUNDARIES OF LEVI-FLAT HYPERSURFACES: SPECIAL HYPERBOLIC POINTS

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ABSTRACT. Let  $S \subset \mathbb{C}^n$ ,  $n \geq 3$  be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is  $S$ , possibly as a current. Our goal is to get examples of such  $S$  containing at least one special 1-hyperbolic point: sphere with two horns; elementary models and their gluing. The particular cases of graphs are also described.

## 1. INTRODUCTION

Let  $S \subset \mathbb{C}^n$ , be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface  $M \subset \mathbb{C}^n \setminus S$  such that  $dM = S$  (i.e. whose boundary is  $S$ , possibly as a current). The case  $n = 2$  has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg; Shcherbina, Chirka, G. Tomassini, Slodkowski, Gromov, Eliashberg; it needs global conditions:  $S$  has to be contained in the boundary of a strictly pseudoconvex domain.

We consider the case  $n \geq 3$ ; results on this case has been obtained since 2005 by Dolbeault, Tomassini and Zaitsev, local necessary conditions recalled in section 2 have to be satisfied by  $S$ , the singular CR points on  $S$  are supposed to be elliptic and the solution  $M$  is obtained in the sense of currents [DTZ05, DTZ10]. More recently a regular solution  $M$  has been obtained when  $S$  satisfies a supplementary global condition as in the case  $n = 2$  [DTZ09], the singular CR points on  $S$  still supposed to be elliptic.

The problem we are interested in is to get examples of such  $S$  containing at least one special 1-hyperbolic point (section 2.4). The CR-orbits near a special 1-hyperbolic point are large and, assuming them compact, a careful examination has to be done (sections 2.6, 2.7). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the numbers of special elliptic and 1-hyperbolic points (section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first considered example is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (section 3.4). Then we consider elementary models and their gluing to obtain more complicated examples (section 3.5). Results have been announced in [Dol08], and

in more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ09] on regularity of the solution when  $S$  is a graph satisfying a supplementary global condition, as in the case  $n = 2$ , to the case of existence of special 1-hyperbolic points, and to gluing of elementary smooth models (section 4).

## 2. PRELIMINARIES: LOCAL AND GLOBAL PROPERTIES OF THE BOUNDARY

**2.1. Definitions.** A smooth, connected, CR submanifold  $M \subset \mathbb{C}^n$  is called *minimal* at a point  $p$  if there does not exist a submanifold  $N$  of  $M$  of lower dimension through  $p$  such that  $HN = HM|_N$ . By a theorem of Sussman, all possible submanifolds  $N$  such that  $HN = HM|_N$  contain, as germs at  $p$ , one of the minimal possible dimension, defining a so called CR *orbit* of  $p$  in  $M$  whose germ at  $p$  is uniquely determined.

Let  $S$  be a smooth compact connected oriented submanifold of dimension  $2n - 2$ .  $S$  is said to be a *locally flat boundary* at a point  $p$  if it locally bounds a Levi-flat hypersurface near  $p$ . Assume that  $S$  is CR in a small enough neighborhood  $U$  of  $p \in S$ . If all CR orbits of  $S$  are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:

- (i)  $S$  is a locally flat boundary on  $U$ ;
- (ii)  $S$  is nowhere minimal on  $U$ .

**2.2. Complex points of  $S$ .** (i.e. singular CR points on  $S$ ) [DTZ05].

At such a point  $p \in S$ ,  $T_p S$  is a complex hyperplane in  $T_p \mathbb{C}^n$ . In suitable local holomorphic coordinates  $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$  vanishing at  $p$ , with  $w = z_n$  and  $z = (z_1, \dots, z_{n-1})$ ,  $S$  is locally given by the equation

$$(1) \quad w = \varphi(z) = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \leq i, j \leq n-1} (a_{ij} z_i z_j + b_{ij} z_i \bar{z}_j + c_{ij} \bar{z}_i \bar{z}_j)$$

$S$  is said *flat* at a complex point  $p \in S$  if  $\sum b_{ij} z_i \bar{z}_j \in \lambda \mathbf{R}, \lambda \in \mathbb{C}$ . We also say that  $p$  is *flat*.

*Let  $S \subset \mathbb{C}^n$  be a locally flat boundary with a complex point  $p$ . Then  $p$  is flat.*

By making the change of coordinates  $(z, w) \mapsto (z, \lambda^{-1} w)$ , we get  $\sum b_{ij} z_i z_j \in \mathbf{R}$  for all  $z$ . By a change of coordinates  $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$  we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form  $Q$  real-valued.

We say that  $S$  is in a *flat normal form* at  $p$  if the coordinates  $(z, w)$  as in (1) are chosen such that  $Q(z) \in \mathbf{R}$  for all  $z \in \mathbb{C}^{n-1}$ .

**2.2.1. Properties of  $Q$ .** Assume that  $S$  is in a flat normal form; then, the quadratic form  $Q$  is real valued. If  $Q$  is positive definite or negative definite, the point  $p \in S$  is said to be *elliptic*; if the point  $p \in S$  is not elliptic, and if  $Q$  is non degenerate,  $p$  is said to be *hyperbolic*. From section 2.4, we will only consider particular cases of the quadratic form  $Q$ .

### 2.3. Elliptic points.

#### 2.3.1. Properties of $Q$ .

**Proposition 1.** ([DTZ05, DTZ10]). *Assume that  $S \subset \mathbb{C}^n$ , ( $n \geq 3$ ) is nowhere minimal at all its CR points and has an elliptic flat complex point  $p$ . Then there exists a neighborhood  $V$  of  $p$  such that  $V \setminus \{p\}$  is foliated by compact real  $(2n - 3)$ -dimensional CR orbits diffeomorphic to the sphere  $\mathbb{S}^{2n-3}$  and there exists a smooth function  $\nu$ , having the CR orbits as the level surfaces.*

*Sketch of Proof.* (see [DTZ10]). In the case of a quadric  $S_0$  ( $w = Q(z)$ ), the CR orbits are defined by  $w_0 = Q(z)$ , where  $w_0$  is constant. Using (1), we approximate the tangent space to  $S$  by the tangent space to  $S_0$  at a point with the same coordinate  $z$ ; the same is done for the tangent spaces to the CR orbits on  $S$  and  $S_0$ ; then we construct the global CR orbit on  $S$  through any given point close enough to  $p$ .  $\square$

**2.4. Special flat complex points.** From [Bis65], for  $n = 2$ , in suitable local holomorphic coordinates centered at 0,  $Q(z) = (z\bar{z} + \lambda \operatorname{Re} z^2)$ ,  $\lambda \geq 0$ , under the notations of [BK91]; for  $0 \leq \lambda < 1$ ,  $p$  is said to be *elliptic*, and for  $1 < \lambda$ , it is said to be *hyperbolic*. The parabolic case  $\lambda = 1$ , not generic, will be omitted [BK91]. When  $n \geq 3$ , the Bishop's reduction cannot be generalized.

We say that the flat complex point  $p \in S$  is *special* if in convenient holomorphic coordinates centered at 0,

$$(2) \quad Q(z) = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \operatorname{Re} z_j^2), \quad \lambda_j \geq 0$$

Let  $z_j = x_j + iy_j$ ,  $x_j, y_j$  real,  $j = 1, \dots, n-1$ , then:

$$(3) \quad Q(z) = \sum_{l=1}^{n-1} ((1 + \lambda_l)x_l^2 + (1 - \lambda_l)y_l^2).$$

A flat point  $p \in S$  is said to be *special elliptic* if  $0 \leq \lambda_j < 1$  for any  $j$ .

A flat point  $p \in S$  is said to be *special  $k$ -hyperbolic* if  $1 < \lambda_j$  for  $j \in J \subset \{1, \dots, n-1\}$  and  $0 \leq \lambda_j < 1$  for  $j \in \{1, \dots, n-1\} \setminus J \neq \emptyset$ , where  $k$  denotes the number of elements of  $J$ .

*Special elliptic (resp. special  $k$ -hyperbolic) points are elliptic (resp. hyperbolic).*

**2.5. Special hyperbolic points.**  $S$  being given by (1), let  $S_0$  be the quadric of equation  $w = Q(z)$ .

**Lemma 2.** *Suppose that  $S_0$  is flat at 0 and that 0 is a special  $k$ -hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates,  $S_0$  is CR and nowhere minimal outside 0, and the CR orbits of  $S_0$  are the  $(2n - 3)$ -dimensional submanifolds given by  $w = \operatorname{const.} \neq 0$ .*

*Proof.* The submanifolds  $w = \text{const.} \neq 0$  have the same complex tangent space as  $S_0$  and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of section 2.1,  $S_0$  is nowhere minimal outside 0.

The section  $w = 0$  of  $S_0$  is a real quadratic cone  $\Sigma'_0$  in  $\mathbf{R}^{2n}$  whose vertex is 0 and, outside 0, it is a CR orbit  $\Sigma_0$  in the neighborhood of 0. We will improperly call  $\Sigma'_0$  a *singular CR orbit*.  $\square$

**2.6. Foliation by CR-orbits in the neighborhood of a special 1-hyperbolic point.** We first mimic and transpose the beginning of the proof of Proposition 1, i.e. of 2.4.2. in ([DTZ05, DTZ09]).

**2.6.1. Local 2-codimensional submanifolds.** In order to use simple notations, we will assume  $n = 3$ .

In  $\mathbb{C}^3$ , consider the 4-dimensional submanifold  $S$  locally defined by the equation

$$(1) \quad w = \varphi(z) = Q(z) + O(|z|^3)$$

and the 4-dimensional submanifold  $S_0$  of equation

$$(4) \quad w = Q(z)$$

with

$$Q = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2$$

having a special 1-hyperbolic point at 0, ( $\lambda_1 > 1, 0 \leq \lambda_2 < 1$ ), and the cone  $\Sigma'_0$  whose equation is:  $Q = 0$ . On  $S_0$ , a CR orbit is the 3-dimensional submanifold  $\mathcal{K}_{w_0}$  whose equation is  $w_0 = Q(z)$ . If  $w_0 > 0$ ,  $\mathcal{K}_{w_0}$  does not cut the line  $L = \{x_1 = x_2 = y_2 = 0\}$ ; if  $w_0 < 0$ ,  $\mathcal{K}_{w_0}$  cuts  $L$  at two points.

**Lemma 3.**  $\Sigma_0 = \Sigma'_0 \setminus 0$  has two connected components in a neighborhood of 0.

*Proof.* The equation of  $\Sigma'_0 \cap \{y_1 = 0\}$  is

$(\lambda_1 + 1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2 = 0$  whose only zero, in the neighborhood of 0, is  $\{0\}$ : the connected components are obtained for  $y_1 > 0$  and  $y_1 < 0$  respectively.  $\square$

**2.6.2. CR-orbits.** By differentiating (1), we get for the tangent spaces the following asymptotics

$$(5) \quad T_{(z, \varphi(z))}S = T_{(z, Q(z))}S_0 + O(|z|^2), \quad z \in \mathbb{C}^2$$

Here both  $T_{(z, \varphi(z))}S$  and  $T_{(z, Q(z))}S_0$  depend continuously on  $z$  near the origin.

Consider

(i) the hyperboloid  $H_- = \{Q = -1\}$ , (then  $Q(\frac{z}{(-Q(z))^{1/2}}) = -1$ ), and the projection:

$$\pi_- : \mathbb{C}^3 \setminus \{z = 0\} \rightarrow H_-, \quad (z, w) \mapsto \frac{z}{(-Q(z))^{1/2}},$$

(ii) for every  $z \in H_-$ , a real orthonormal basis  $e_1(z), \dots, e_6(z)$  of  $\mathbb{C}^3 \cong \mathbb{R}^6$  such that

$$e_1(z), e_2(z) \in H_z H_-, \quad e_3(z) \in T_z H_-,$$

where  $HH_-$  is the complex tangent bundle to  $H_-$ .

Locally such a basis can be chosen continuously depending on  $z$ . For every  $(z, w) \in \mathbb{C}^3 \setminus \{z = 0\}$ , consider the basis  $e_1(\pi_-(z, w)), \dots, e_6(\pi_-(z, w))$ . The unit vectors  $e_1(\pi_-(z, w_0)), e_2(\pi_-(z, w_0)), e_3(\pi_-(z, w_0))$  are tangent to the CR orbit  $\mathcal{K}_{w_0}$  in  $(z, w_0)$  for  $w_0 < 0$ . Then, from (5), we have:

$$(6) \quad H_{(z, \varphi(z))} S = H_{(z, Q(z))} S_0 + O(|z|^2), \quad z \neq 0, \quad z \rightarrow 0.$$

As in [DTZ10], in the neighborhood of 0, denote by  $E(q), q \in S \setminus \{0\}, w < 0$  the tangent space to the local CR orbit  $\mathcal{K}$  on  $S$  through  $q$ , and by  $E_0(q_0), q_0 \in S_0 \setminus \{0\}, w < 0$  the analogous object for  $S_0$ . We have :

$$(7) \quad E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \quad z \neq 0, \quad z \rightarrow 0$$

Given  $\underline{q} \in S$ , by integration of  $E(q), q \in S$ , we get, locally, the CR orbit (the leaf), on  $S$  through  $\underline{q}$ ; given  $\underline{q}_0 \in S_0$ , by integration of  $E_0(q_0), q_0 \in S_0$ , we get, locally, the CR orbit (the leaf), on  $S_0$  through  $\underline{q}_0$  (theorem of Sussman). On  $S_0$ , a leaf is the 3-dimensional submanifold  $\mathcal{K}_{\underline{q}_0} = \mathcal{K}_{w_0} = \mathcal{K}_0$  whose equation is  $w_0 = Q(z)$ , with  $\underline{q} = (z_0, w_0 = Q(z_0))$ .  $d\pi_-$  projects each  $E_0(q), q \in S_0, w < 0$ , bijectively onto  $T_{\pi(q)} H_-$ , then  $\pi_-|_{\mathcal{K}_0}$  is a diffeomorphism onto  $H_-$ ; this implies, from (7), that, in a suitable neighborhood of the origin, the restriction of  $\pi_-$  to each local CR orbit of  $S$  is a local diffeomorphism.

We have:  $\varphi(z) = Q(z) + \Phi(z)$  with  $\Phi(z) = O(|z|^3)$ .

**2.6.3. Behaviour of local CR orbits.** Follow the construction of  $E(z, \varphi(z))$ ; compare with  $E_0(z, Q(z))$ . We know the integral manifold, the orbit of  $E_0(z, Q(z))$ ; deduce an evaluation of the integral manifold  $\mathcal{K}$  of  $E(z, \varphi(z))$ .

**Lemma 4.** *Under the above hypotheses, the local orbit  $\Sigma$  corresponding to  $\Sigma_0$  has two connected components in the neighborhood of 0.*

*Proof.* Using the real coordinates, as for Lemma 3, consider  $\Sigma' \cap \{y_1 = 0\}$ . Locally, the connected components are obtained for  $y_1 > 0$  and  $y_1 < 0$  respectively, from formula (1).  $\square$

We will improperly call  $\Sigma' = \overline{\Sigma}$  a *singular CR orbit* and a *singular leaf of the foliation*.

- We intend to prove: 1)  $\mathcal{K}$  does not cross the singular leaf through 0;  
2) the only separatrix is the singular leaf through 0.

From the orbit  $\mathcal{K}_0$ , construct the differential equation defining it, and using (7), construct the differential equation defining  $\mathcal{K}$ .

In  $\mathbb{C}^3$ , we use the notations:  $x = x_1, y = y_1, u = x_2, v = y_2$ ; it suffices to consider the particular case:  $Q = 3x^2 - y^2 + u^2 + v^2$ . On  $S_0$ , the orbit  $\mathcal{K}_0$

issued from the point  $(c, 0, 0, 0)$  is defined by:  $3x^2 - y^2 - u^2 + v^2 = 3c^2$ , i.e., for  $x \geq 0$ ,  $x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} = A(y, u, v)$ ; the local coordinates on the orbit are  $(y, u, v)$ .  $\mathcal{K}_0$  satisfies the differential equation:  $dx = dA$ . From (7), the orbit  $\mathcal{K}$ , issued from  $(c, 0, 0, 0)$ , satisfies  $dx = dA + \Psi$  with  $\Psi(y, u, v; c) = O(|z|^2)$ ; hence  $\Psi = d\Phi$ , then  $x = A + \Phi$ , with  $\Phi = O(|z|^3)$ . More explicitly,  $\mathcal{K}$  is defined by:

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} + \Phi(y, u, v; c), \quad \Phi(y, u, v; c) = O(|z|^3)$$

The cone  $\Sigma'_0$  whose equation is:  $Q = 0$  is a separatrix for the orbits  $\mathcal{K}_0$ . The corresponding object  $\Sigma' = \{\varphi(z) = 0\}$  for  $S$  has the singular point 0 and for  $x > 0, y > 0, u > 0, v > 0$  is defined by the differential equation  $dx = d(A + \Phi)$ , with  $c = 0$ , i.e. the local equation of  $\Sigma'$  is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{\frac{1}{2}} + \Phi(y, u, v; 0), \quad \Phi(y, u, v; 0) = O(|z|^3)$$

For given  $(y, u, v)$ ,  $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$ . But  $x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(1)$  and  $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$ .

As a consequence, for  $x > 0, y > 0, u > 0, v > 0$ , locally,  $\Sigma'$  is a separatrix for the orbits  $\mathcal{K}$ , and the only one. Same result for  $x < 0$ .

2.6.4. What has been done from the hyperboloïd  $H_- = \{Q = -1\}$  can be repeated from the hyperboloïd  $H_+ = \{Q = 1\}$ .

As at the beginning of the section 2.6.2, we consider

(i) the hyperboloïd  $H_+\{Q = 1\}$  and the projection:

$$\pi_+ : \mathbb{C}^3 \setminus \{z = 0\} \rightarrow H_+, \quad (z, w) \mapsto \frac{z}{(Q(z))^{1/2}},$$

(ii) for every  $z \in H_+$ , a real orthonormal basis  $e_1(z), \dots, e_6(z)$  of  $\mathbb{C}^3 \cong \mathbb{R}^6$  such that

$$e_1(z), e_2(z) \in H_z H_+, \quad e_3(z) \in T_z H_+,$$

where  $HH_+$  is the complex tangent bundle to  $H_+$ .

2.6.5.

**Lemma 5.** *Given  $\varphi$ , there exists  $R > 0$  such that, in  $B(0, R) \cap \{x > 0, y > 0, u > 0, v > 0\} \subset \mathbb{C}^2$ , the CR orbits  $\mathcal{K}$  have  $\Sigma'$  as unique separatrix.*

*Proof.* When  $c$  tends to zero,  $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(|z|)$ ,  $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$ . For  $\varphi(z) = Q(z) + \Phi(z)$  with  $\Phi(z) = O(|z|^3)$  given, in (7),  $E(z, \varphi(z)) - E_0(z, Q(z)) = O(|z|^2)$  and  $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$  are also given. Then there exists  $R$  such that, for  $|z| < R$ ,  $x_{\mathcal{K},c} - x_{\mathcal{K},0} > 0$ .  $\square$

**2.7. CR-orbits near a subvariety containing a special 1-hyperbolic point.**

2.7.1. In the section 2.7, we will impose conditions on  $S$  and give a local property in the neighborhood of a compact  $(2n - 3)$ -subvariety of  $S$ .

Assume that  $S \subset \mathbb{C}^n$  ( $n \geq 3$ ), is a locally closed  $(2n - 2)$ -submanifold, nowhere minimal at all its CR points, which has a unique 1-hyperbolic flat complex point  $p$ , and such that:

(i)  $\Sigma$  being the orbit whose closure  $\Sigma'$  contains  $p$ , then  $\Sigma'$  is compact.

Let  $q \in S$ ,  $q \neq p$ ; then, in a neighborhood  $U$  of  $q$  disjoint from  $p$ ,  $S$  is CR,  $\text{CR-dim } S = n - 2$ ,  $S$  is non minimal and  $\Sigma$  is 1-codimensional. To show that the CR orbits constitute a foliation on  $S$  whose separatrix is  $\Sigma'$ : this is true in  $U$  since  $\Sigma \cap U$  is a leaf. Moreover, let  $U_0$  the ball  $B(0, R)$  centered in  $p = 0$  in Lemma 5, if  $U \cap U_0 \neq \emptyset$ , the leaves in  $U$  glue with the leaves in  $U_0$  on  $U \cap U_0$ . Since  $\Sigma'$  is compact, there exists a finite number of points  $q_j \in \Sigma'$ ,  $j = 0, 1, \dots, J$ , and open neighborhoods  $U_j$ , as above, such that  $(U_j)_{j=0}^J$  is an open covering of  $\Sigma'$ . Moreover the leaves on  $U_j$  glue respectively with the leaves on  $U_k$  if  $U_j \cap U_k \neq \emptyset$ .

2.7.2.

**Proposition 6.** *Assume that  $S \subset \mathbb{C}^n$  ( $n \geq 3$ ), is a locally closed  $(2n - 2)$ -submanifold, nowhere minimal at all its CR points, which has a unique special 1-hyperbolic flat complex point  $p$ , and such that:*

(i)  $\Sigma$  being the orbit whose closure  $\Sigma'$  contains  $p$ , then  $\Sigma'$  is compact;

(ii)  $\Sigma$  has two connected components  $\sigma_1, \sigma_2$ , whose closures are homeomorphic to spheres of dimension  $2n - 3$ .

*Then, there exists a neighborhood  $V$  of  $\Sigma'$  such that  $V \setminus \Sigma'$  is foliated by compact real  $(2n - 3)$ -dimensional CR orbits whose equation, in a neighborhood of  $p$  is (3), and, the  $w(= x_n)$ -axis being assumed to be vertical, each orbit is diffeomorphic to*

*the sphere  $\mathbf{S}^{2n-3}$  above  $\Sigma'$ ,*

*the union of two spheres  $\mathbf{S}^{2n-3}$  under  $\Sigma'$ ,*

*and there exists a smooth function  $\nu$ , having the CR orbits as the level surfaces.*

*Proof.* From subsection 2.7.1 and the following remark:

When  $x_n$  tends to 0, the orbits tends to  $\Sigma'$ , and because of the geometry of the orbits near  $p$ , they are diffeomorphic to a sphere above  $\Sigma'$ , and to the union of two spheres under  $\Sigma'$ . The existence of  $\nu$  is proved as in Proposition 1, namely, consider a smooth curve  $\gamma : [0, \varepsilon) \rightarrow S$  such that  $\gamma(0) = p$ , and  $\gamma$  is a diffeomorphism onto its image  $\Gamma = \gamma([0, \varepsilon))$ . Let  $\nu = \gamma^{-1}$  on the image of  $\gamma$ , then, close enough to  $p$ , every CR orbit cuts  $\gamma([0, \varepsilon))$  at a unique point  $q(t)$ ,  $t \in [0, \varepsilon)$ . Hence there is a unique extension of  $\nu$  from  $\gamma([0, \varepsilon))$  to a neighborhood of  $\Sigma'$  having CR orbits as its level surfaces.  $\nu$  being smooth away from  $p$ , it is smooth on the orbit  $\Sigma$  and on a neighborhood of  $\Sigma \cup \{p\} = \Sigma'$ .  $\square$

**2.8. Geometry of the complex points of  $S$ .** The results of section 2.8 are particular cases of theorems of H-F Lai [Lai72], that I learnt from F. Forstneric in July 2011.

In [BK91] E. Bedford & W. Klingenberg cite the following theorem of E. Bishop [Bis65][section 4, p.15]: *On a 2-sphere embedded in  $\mathbb{C}^2$ , the difference between the numbers of elliptic points and of hyperbolic points is the Euler-Poincaré characteristic, i.e. 2.* For the proof, Bishop uses a theorem of ([CS 51], section 4).

We extend the result for  $n \geq 3$  and give proofs which are essentially the same than in the general case of [Lai72, Lai74] but simpler.

**2.8.1.** Let  $S$  be a smooth compact connected oriented submanifold of dimension  $2n - 2$ . Let  $G$  be the manifold of the oriented real linear  $(2n - 2)$ -subspaces of  $\mathbb{C}^n$ . The submanifold  $S$  of  $\mathbb{C}^n$  has a given orientation which defines an orientation  $o(p)$  of the tangent space to  $S$  at any point  $p \in S$ . By mapping each point of  $S$  into its oriented tangent space, we get a smooth Gauss map

$$t : S \rightarrow G$$

Denote  $-t(p)$  the tangent space to  $S$  at  $p$  with opposite orientation  $-o(p)$ .

**2.8.2. Properties of  $G$ .** (a)  $\dim G = 2(2n - 2)$ .

*Proof.*  $G$  is a two-fold covering of the Grassmannian  $M_{m,k}$ , of the linear  $k$ -subspaces of  $\mathbb{R}^m$  [Ste99][Part, section 7.9], for  $m = 2n$  and  $k = 2n - 2$ ; they have the same dimension. We have:

$$M_{m,k} \cong O_m/O_k \times O_{m-k}$$

But  $\dim O_k = \frac{1}{2}k(k - 1)$ , hence  $\dim M_{m,k} = \frac{1}{2}(m(m - 1) - k(k - 1) - (m - k)(m - k - 1)) = k(m - k)$ .

(b)  $G$  has the complex structure of a smooth quadric of complex dimension  $(2n - 2)$  of  $\mathbb{C}P^{2n-1}$  [Lai74], [Pol08].

(c) There exists a canonical isomorphism  $h : G \rightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$ .

(d) Homology of  $G$  (cf [Pol08]): Let  $S_1, S_2$  be generators of  $H_{2n-2}(G, \mathbb{Z})$ ; we assume that  $S_1$  and  $S_2$  are fundamental cycles of complex projective subspaces of complex dimension  $(n - 1)$  of the complex quadric  $G$ . We also denote  $S_1, S_2$  the ordered two factors  $\mathbb{C}P^{n-1}$ , so that  $h : G \rightarrow S_1 \times S_2$ .  $\square$

**2.8.3.**

**Proposition 7.** *For  $n \geq 2$ , in general,  $S$  has isolated complex points.*

*Proof.* Let  $\pi \in G$  be a complex hyperplane of  $\mathbb{C}^n$  whose orientation is induced by its complex structure; the set of such  $\pi$  is  $H = G_{n-1,n}^{\mathbb{C}} = \mathbb{C}P^{n-1*} \subset G$ , as real submanifold. If  $p$  is a complex point of  $S$ , then  $t(p) \in H$  or  $-t(p) \in H$ . The set of complex points of  $S$  is the inverse image by  $t$  of



the intersections  $t(S) \cap H$  and  $-t(S) \cap H$  in  $G$ . Since  $\dim t(S) = 2n - 2$ ,  $\dim H = 2(n - 1)$ ,  $\dim G = 2(2n - 2)$ , the intersection is 0-dimensional, in general.  $\square$

2.8.4. Denoting also  $S$ , the fundamental cycle of the submanifold  $S$  and  $t_*$  the homomorphism defined by  $t$ , we have:

$$t_*(S) \sim u_1 S_1 + u_2 S_2$$

where  $\sim$  means *homologous to*.

2.8.5.

**Lemma 8** (proved for  $n = 2$  in [CS51]). *With the above notations, we have:  $u_1 = u_2$ ;  $u_1 + u_2 = \chi(S)$ , Euler-Poincaré characteristic of  $S$ .*

The proof for  $n = 2$  works for any  $n \geq 3$ , namely:

Let  $G'$  be the manifold of the oriented real linear 2-subspaces of  $\mathbb{C}^n$ . Let  $\alpha : G \rightarrow G'$  map each oriented  $2(n - 1)$ -subspace  $R$  onto its normal 2-subspace  $R'$  oriented so that  $R, R'$  determine the orientation of  $\mathbb{C}^n$ .  $\alpha$  is a canonical isomorphism. Let  $n : S \rightarrow G'$  the map defined by taking oriented normal planes; then:  $n = \alpha t$  and  $t = \alpha^{-1} n$ , hence the mapping  $h\alpha h^{-1} : S_1 \times S_2 \rightarrow S_1 \times S_2$ . Let  $(x, y)$  be a point of  $S_1 \times S_2$ , then  $(\dagger) \ h\alpha h^{-1}(x, y) = (x, -y)$ .

Over  $G$ , there is a bundle  $V$  of spheres obtained by considering as fiber over a real oriented linear  $(2n - 2)$ -subspace of  $\mathbb{C}^n$  through 0 the unit sphere  $S^{2n-3}$  of this subspace. Let  $\Omega$  be the characteristic class of  $V$ , and let  $\Omega_t$ ,  $\Omega_n$  denote the characteristic classes of the tangent and normal bundles of  $S$ . Then  $t^* \Omega = \Omega_t$ ,  $n^* \Omega = \Omega_n$ .

$V$  is the Stiefel manifold of ordered pairs of orthogonal unit vectors through in  $\mathbb{R}^{2n} \cong \mathbb{C}^n$ . Let  $f : V \rightarrow G$  the projection.

From the Gysin sequence, we see that the kernel of  $f^* : H^{2n-2}(G) \rightarrow H^{2n-2}(V)$  is generated by  $\Omega$ . To find the kernel of  $f^*$ , we determine the morphism  $f_* : H_{2n-2}(V) \rightarrow H_{2n-2}(G)$ . A generating  $2n - 2$ -cycle of in  $V$  is  $S^2 \times e$  where  $S^2 \cong \mathbb{C}P^{n-1}$  and  $e$  is a point. Let  $z$  be any point of  $S^2$ , then from  $(\dagger)$ , we have

$$hf(z, e) = (z, -z)$$

Therefore, we see that  $f_*(S^2 \times e) = S_1 - S_2$ . Then, the kernel of  $f^*$  is  $\mathbb{Z}$ -generated by  $S_1^* + S_2^*$ .

With convenient orientation for the fibre of the bundle  $V$ , we get:  $\Omega = S_1^* + S_2^*$ . For convenient orientation of  $S$ , we get  $\Omega_t.S = \chi_S =$  Euler characteristic of  $S$ . We have

$$\Omega_t = t^*(S_1^* + S_2^*) = t^*S_1^* + t^*S_2^*$$

$$\Omega_n = n^*(S_1^* + S_2^*) = t^*\alpha^*(S_1^* + S_2^*) = t^*(S_1^* - S_2^*) = t^*S_1^* - t^*S_2^*$$

Since  $\Omega_n = 0$ , we get:

$$(t^* S_1^*).S = (t^* S_2^*).S = \frac{1}{2}\chi_S$$

**2.8.6. Local intersection numbers of  $H$  and  $t(S)$  when all complex points are flat and special.**  $H$  is a complex linear  $(n-1)$ -subspace of  $G$ , then is homologous to one of the  $S_j$ ,  $j = 1, 2$ , say  $S_2$  when  $G$  has its structure of complex quadric. The intersection number of  $H$  and  $S_1$  is 1 and the intersection number of  $H$  and  $S_2$  is 0. So, the intersection number of  $H$  and  $u_1 S_1 + u_2 S_2$  is  $u_1$ .

In the neighborhood of a complex point 0,  $S$  is defined by equation (1), with  $w = z_n$  and

$$(1') \quad Q(z) = \sum_{j=1}^{n-1} \mu_j (z_j \bar{z}_j + \lambda_j \operatorname{Re} z_j^2), \quad \mu_j > 0, \lambda_j \geq 0$$

Let  $z_j = x_{2j-1} + ix_{2j}$ ,  $j = 1, \dots, n$ , with real  $x_l$ . Let  $e_l$  the unit vector of the  $x_l$  axis,  $l = 1, \dots, 2n$ .

For simplicity assume  $n = 3$ :  $Q(z) = \mu_1(z_1 \bar{z}_1 + \lambda_1 \operatorname{Re} z_1^2) + \mu_2(z_2 \bar{z}_2 + \lambda_2 \operatorname{Re} z_2^2)$ , with  $\mu_1 = \mu_2 = 1$ .

Then, up to higher order terms,  $S$  is defined by:

$$z_1 = x_1 + ix_2; \quad z_2 = x_3 + ix_4; \quad z_3 = (1 + \lambda_1)x_1^2 + (1 - \lambda_1)x_2^2 + (1 + \lambda_2)x_3^2 + (1 - \lambda_2)x_4^2.$$

In the neighborhood of 0, the tangent space to  $S$  is defined by the four independent vectors

$$\begin{aligned} \nu_1 &= e_1 + 2(1 + \lambda_1)x_1 e_5; \quad \nu_2 = e_2 + 2(1 - \lambda_1)x_2 e_5; \quad \nu_3 = e_3 + 2(1 + \lambda_2)x_3 e_5; \\ \nu_4 &= e_4 + 2(1 - \lambda_2)x_4 e_5 \end{aligned}$$

Then, if 0 is special elliptic or special  $k$ -hyperbolic with  $k$  even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special  $k$ -hyperbolic with  $k$  odd the tangent space has opposite orientation.

**2.8.7.**

**Proposition 9** (known for  $n = 2$  [Bis65], here for  $n \geq 3$ ). *Let  $S$  be a smooth, oriented, compact, 2-codimensional, real submanifold of  $\mathbb{C}^n$  whose all complex points are flat and special elliptic or special 1-hyperbolic. Then, on  $S$ ,  $\sharp$  (special elliptic points) -  $\sharp$  (special 1-hyperbolic points) =  $\chi(S)$ . If  $S$  is a sphere, this number is 2.*

*Proof.* Let  $p \in S$  be a complex point and  $\pi$  be the tangent hyperplane to  $S$  at  $p$ . Assume that

(\*\*) *the orientation of  $S$  induces, on  $\pi$ , the orientation given by its complex structure,*

then  $\pi \in H$ .

If  $p$  is elliptic, the intersection number of  $H$  and  $t(S)$  is 1; if  $p$  is 1-hyperbolic, the intersection number of  $H$  and  $t(S)$  is -1 at  $p$ .

From the beginning of section 2.8.6, the sum of the intersection numbers of  $H$  and  $t(S)$  at complex points  $p$  satisfying  $(**)$  is  $u_1$ . Reversing the condition  $(**)$ , and using Lemma 8, we get the Proposition.  $\square$

### 3. PARTICULAR CASES: HORNED SPHERE; ELEMENTARY MODELS AND THEIR GLUING

3.1. We recall the following Harvey-Lawson theorem with real parameter to be used later.

3.1.1. Let  $E \cong \mathbf{R} \times \mathbb{C}^n$ , and  $k : \mathbf{R} \times \mathbb{C}^n \rightarrow \mathbf{R}$  be the projection. Let  $N \subset E$  be a compact, (oriented) CR subvariety of  $\mathbb{C}^{n+1}$  of real dimension  $2n - 2$  and CR dimension  $n - 2$ , ( $n \geq 3$ ), of class  $C^\infty$ , with negligible singularities (i.e. there exists a closed subset  $\tau \subset N$  of  $(2n - 2)$ -dimensional Hausdorff measure 0 such that  $N \setminus \tau$  is a CR submanifold). Let  $\tau'$  be the set of all points  $z \in N$  such that either  $z \in \tau$  or  $z \in N \setminus \tau$  and  $N$  is not transversal to the complex hyperplane  $k^{-1}(k(z))$  at  $z$ . Assume that  $N$ , as a current of integration, is  $d$ -closed and satisfies:

(H) there exists a closed subset  $L \subset \mathbb{R}_{x_1}$  with  $H^1(L) = 0$  such that for every  $x \in k(N) \setminus L$ , the fiber  $k^{-1}(x) \cap N$  is connected and does not intersect  $\tau'$ .

3.1.2.

**Theorem 10** ([DTZ10] (see also [DTZ05])). *Let  $N$  satisfy (H) with  $L$  chosen accordingly. Then, there exists, in  $E' = E \setminus k^{-1}(L)$ , a unique  $C^\infty$  Levi-flat  $(2n - 1)$ -subvariety  $M$  with negligible singularities in  $E' \setminus N$ , foliated by complex  $(n - 1)$ -subvarieties, with the properties that  $M$  simply (or trivially) extends to  $E'$  as a  $(2n - 1)$ -current (still denoted  $M$ ) such that  $dM = N$  in  $E'$ . The leaves are the sections by the hyperplanes  $E_{x_1^0}$ ,  $x_1^0 \in k(N) \setminus L$ , and are the solutions of the “Harvey-Lawson problem” for finding a holomorphic subvariety in  $E_{x_1^0} \cong \mathbb{C}^n$  with prescribed boundary  $N \cap E_{x_1^0}$ .*

3.1.3.

**Remark 11.** *Theorem 10 is valid in the space  $E \cap \{\alpha_1 < x_1 < \alpha_2\}$ , with the corresponding condition (H). Moreover, since  $N$  is compact, for convenient coordinate  $x_1$ , we can assume  $x_1 \in [0, 1]$ .*

3.2. *To solve the boundary problem by Levi-flat hypersurfaces,  $S$  has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.*

3.3. **Sphere with one special 1-hyperbolic point (sphere with two horns): Example.**

3.3.1. In  $\mathbb{C}^3$ , let  $(z_j)$ ,  $j = 1, 2, 3$ , be the complex coordinates and  $z_j = x_j + iy_j$ . In  $\mathbf{R}^6 \cong \mathbb{C}^3$ , consider the 4-dimensional subvariety (with negligible singularities)  $S$  defined by:

$$\begin{aligned} y_3 &= 0 \\ 0 \leq x_3 \leq 1; \quad &x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + \\ &4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0 \\ -1 \leq x_3 \leq 0; \quad &x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 \end{aligned}$$

The singular set of  $S$  is the 3-dimensional section  $x_3 = 0$  along which the tangent space is not everywhere (uniquely) defined.  $S$  being in the real hyperplane  $\{y_3 = 0\}$ , the complex tangent spaces to  $S$  are  $\{x_3 = x^0\}$  for convenient  $x^0$ .

3.3.2. The tangent space to the hypersurface  $f(x_1, y_1, x_2, y_2, x_3) = 0$  in  $\mathbb{R}^5$  is

$$X_1 f'_{x_1} + Y_1 f'_{y_1} + X_2 f'_{x_2} + Y_2 f'_{y_2} + X_3 f'_{x_3} = 0,$$

Then, the tangent space to  $S$  in the hyperplane  $\{y_3 = 0\}$  is:

for  $0 \leq x_3$ ,

$$\begin{aligned} &2x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1 \\ &+ 2x_2[x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2[x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2 \\ &+ [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1) \\ &\quad - (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0; \end{aligned}$$

for  $x_3 \leq 0$ ,

$$4(x_1^2 + 2)x_1X_1 + 4(y_1^2 - 1)y_1Y_1 + 2(2x_2^2 + 1)x_2X_2 + 2(2y_2^2 + 1)y_2Y_2 - X_3 = 0.$$

3.3.3. The complex points of  $S$  are defined by the vanishing of the coefficients of  $X_j$ ,  $j=1,2,3,4$  in the equation of the tangent spaces

for  $0 \leq x_3 \leq 1$ ,

$$\begin{aligned} x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] &= 0, \\ y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] &= 0, \\ x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] &= 0, \\ y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] &= 0. \end{aligned}$$

We have the solutions

$h$ :  $x_j = 0, y_j = 0$ , ( $j = 1, 2$ ),  $x_3 = 0$ ;

$e_3$ :  $x_j = 0, y_j = 0$ , ( $j = 1, 2$ ),  $x_3 = 1$ .

for  $x_3 \leq 0$ ,

$$\begin{aligned} (x_1^2 + 2)x_1 &= 0, \\ (y_1^2 - 1)y_1 &= 0, \\ (2x_2^2 + 1)x_2 &= 0, \\ (2y_2^2 + 1)y_2 &= 0. \end{aligned}$$

We have the solutions

$$\begin{aligned} h: & x_j = 0, y_j = 0, (j = 1, 2), x_3 = 0; \\ e_1, e_2: & x_1 = 0, y_1 = \pm 1, x_2 = 0, y_2 = 0, x_3 = -1. \end{aligned}$$

Remark that the tangent space to  $S$  at  $h$  is well defined. Moreover, the set  $S$  will be smoothed along its section by the hyperplane  $\{x_3 = 0\}$  by a small deformation leaving  $h$  unchanged. *In the following  $S$  will denote this smooth submanifold.*

### 3.3.4.

**Lemma 12.** *The points  $e_1, e_2, e_3$  are special elliptic; the point  $h$  is special  $\{1\}$ -hyperbolic.*

*Proof.* Point  $e_3$ : Let  $x'_3 = 1 - x_3$ , then the equation of  $S$  in the neighborhood of  $e_3$  is:

$$\begin{aligned} (1 - x'_3)(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3'^2 - 2x'_3) - x'_3(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0, \text{ i.e.} \\ 2x'_3 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + O(|z|^3), \text{ or } w = z\bar{z} + O(|z|^3) \end{aligned}$$

then  $e_3$  is special elliptic.

Points  $e_1, e_2$ : Let  $y'_1 = y_1 \pm 1$ ,  $x'_3 = x_3 + 1$ , then the equation of  $S$  in the neighborhood of  $e_1, e_2$  is:

$$\begin{aligned} x'_3 - 1 = x_1^4 + (y'_1 \mp 1)^4 + x_2^4 + y_2^4 + 4x_1^2 - 2(y'_1 \mp 1)^2 + x_2^2 + y_2^2 \\ = x_1^4 + y_1'^4 \mp 4y_1'^3 + 6y_1'^2 \mp 4y_1' + 1 + x_2^4 + y_2^4 + 4x_1^2 - 2(y'_1 \mp 1)^2 + x_2^2 + y_2^2, \end{aligned}$$

then

$$x'_3 = x_1^4 + y_1'^4 \mp 4y_1'^3 + 4y_1'^2 + x_2^4 + y_2^4 + 4x_1^2 + x_2^2 + y_2^2, \text{ i.e.}$$

$$x'_3 = 4x_1^2 + 4y_1'^2 + x_2^2 + y_2^2 + O(|z|^3), \text{ or } w = 4z_1\bar{z}_1 + z_2\bar{z}_2,$$

then  $e_1, e_2$  are special elliptic.

Point  $h$ : The equation of  $S$  in the neighborhood of  $h$  is:

for  $x_3 \geq 0$ ,

$$\begin{aligned} x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) \\ + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0 \end{aligned}$$

for  $x_3 \leq 0$ ,

$$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2, \text{ i.e.}$$

$x_3 = 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 + O(|z|^3)$ , in both cases, up to the third order terms, i.e.:  $w = z_1\bar{z}_1 + z_2\bar{z}_2 + 3\mathcal{R}e \ z_1^2$ ,

then  $h$  is special  $\{1\}$ -hyperbolic.  $\square$

3.3.5. *Section  $\Sigma' = S \cap \{x_3 = 0\}$ .* Up to a small smooth deformation, its equation is:

$$x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0, \text{ in } \{x_3 = 0\}.$$

The tangent cone to  $\Sigma'$  at 0 is:  $4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$ .

Locally, the section of  $S$  by the coordinate 3-space

$$x_1, y_1, x_3 \text{ is: } x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3)$$

$x_2, y_2, x_3$  is:  $x_3 = x_2^2 + y_2^2 + O(|z|^3)$

3.3.1'. *Shape of  $\Sigma' = S \cap \{x_3 = 0\}$  in the neighborhood of the origin 0 of  $\mathbb{C}^3$ .*

**Lemma 13.** *Under the above hypotheses and notations,*

(i)  $\Sigma = \Sigma' \setminus 0$  has two connected components  $\sigma_1, \sigma_2$ .

(ii) *The closures of the three connected components of  $S \setminus \Sigma'$  are submanifolds with boundaries and corners.*

*Proof.* (i) The only singular point of  $\Sigma'$  is 0. We work in the ball  $B(0, A)$  of  $\mathbb{C}^2$   $(x_1, y_1, x_2, y_2)$  for small  $A$  and in the 3-space  $\pi_\lambda = \{y_2 = \lambda x_2\}$ ,  $\lambda \in \mathbb{R}$ . For  $\lambda$  fixed,  $\pi_\lambda \cong \mathbb{R}^3(x_1, y_1, x_2)$ , and  $\Sigma' \cap \pi_\lambda$  is the cone of equation  $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$  with vertex 0 and basis in the plane  $x_2 = x_2^0$  the hyperboloid  $H_\lambda$  of equation  $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ ; the curves  $H_\lambda$  have no common point outside 0. So, when  $\lambda$  varies, the surfaces  $\Sigma' \cap \pi_\lambda$  are disjoint outside 0. The set  $\Sigma'$  is clearly connected;  $\Sigma' \cap \{y_1 = 0\} = \{0\}$ , the origin of  $\mathbb{C}^3$ ; from above:  $\sigma_1 = \Sigma \cap \{y_1 > 0\}$ ;  $\sigma_2 = \Sigma \cap \{y_1 < 0\}$ .

(ii) The three connected components of  $S \setminus \Sigma'$  are the components which contain, respectively  $e_1, e_2, e_3$  and whose boundaries are  $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_1 \cup \bar{\sigma}_2$ ; these boundaries have corners as shown in the first part of the proof.  $\square$

The connected component of  $\mathbb{C}^2 \times \mathbb{R} \setminus S$  containing the point  $(0, 0, 0, 0, 1/2)$  is the Levi-flat solution, the complex leaves being the sections by the hyperplanes  $x_3 = x_3^0, -1 < x_3^0 < 1$ .

The sections by the hyperplanes  $x_3 = x_3^0$  are diffeomorphic to a 3-sphere for  $0 < x_3^0 < 1$  and to the union of two disjoint 3-spheres for  $-1 < x_3^0 < 0$ , as can be shown intersecting  $S$  by lines through the origin in the hyperplane  $x_3 = x_3^0$ ;  $\Sigma'$  is homeomorphic to the union of two 3-spheres with a common point.

**3.4. Sphere with one special 1-hyperbolic point (sphere with two horns).** The example of section 3.3 shows that the necessary conditions of section 2 can be realised. Moreover, from Proposition 2.8.7, the hypothesis on the number of complex points is meaningful.

3.4.1.

**Proposition 14.** *[cf [Dol08][Proposition 2.6.1]] Let  $S \subset \mathbb{C}^n$  be a compact connected real 2-codimensional manifold such that the following holds:*

- (i)  *$S$  is a topological sphere;  $S$  is nonminimal at every CR point;*
- (ii) *every complex point of  $S$  is flat; there exist three special elliptic points  $e_j, j = 1, 2, 3$  and one special 1-hyperbolic point  $h$ ;*
- (iii)  *$S$  does not contain complex manifolds of dimension  $(n - 2)$ ;*
- (iv) *the singular CR orbit  $\Sigma'$  through  $h$  on  $S$  is compact and  $\Sigma' \setminus \{h\}$  has two connected components  $\sigma_1$  and  $\sigma_2$  whose closures are homeomorphic to spheres of dimension  $2n - 3$ ;*
- (v) *the closures  $S_1, S_2, S_3$  of the three connected components  $S'_1, S'_2, S'_3$  of  $S \setminus \Sigma'$  are submanifolds with (singular) boundary.*

Then each  $S_j \setminus \{e_j \cup \Sigma'\}$ ,  $j = 1, 2, 3$  carries a foliation  $\mathcal{F}_j$  of class  $C^\infty$  with 1-codimensional CR orbits as compact leaves.

*Proof.* From conditions (i) and (ii),  $S$  satisfying the hypotheses of Proposition 1, near any elliptic flat point  $e_j$ , and of Proposition 6 near  $\Sigma'$ , all CR orbits being diffeomorphic to the sphere  $\mathbf{S}^{2n-3}$ . The assumption (iii) guarantees that all CR orbits in  $S$  must be of real dimension  $2n - 3$ . Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of  $\Sigma'$ , we obtain, from  $S \setminus \Sigma'$ , three compact manifolds  $S_j''$ ,  $j = 1, 2, 3$ , with boundary and with the foliation  $\mathcal{F}_j$  of codimension 1 given by its CR orbits whose first cohomology group with values in  $\mathbf{R}$  is 0, near  $e_j$ . It is easy to show that this foliation is transversely oriented.  $\square$

3.4.2. Recall the Thurston's Stability Theorem ([CaC], Theorem 6.2.1).

**Proposition 15.** *Let  $(M, \mathcal{F})$  be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class  $C^1$ . If there is a compact leaf  $L$  with  $H^1(L, \mathbf{R}) = 0$ , then every leaf is homeomorphic to  $L$  and  $M$  is homeomorphic to  $L \times [0, 1]$ , foliated as a product,*

Then, from the above theorem,  $S_j''$  is homeomorphic to  $\mathbf{S}^{2n-3} \times [0, 1]$  with CR orbits being of the form  $\mathbf{S}^{2n-3} \times \{x\}$  for  $x \in [0, 1]$ . Then the full manifold  $S_j$  is homeomorphic to a half-sphere supported by  $\mathbf{S}^{2n-2}$  and  $\mathcal{F}_j$  extends to  $S_j$ ;  $S_3$  having its boundary pinched at the point  $h$ .  $\square$

3.4.3.

**Theorem 16.** *Let  $S \subset \mathbb{C}^n$ ,  $n \geq 3$ , be a compact connected smooth real 2-codimensional submanifold satisfying the conditions (i) to (v) of Proposition 14. Then there exists a Levi-flat  $(2n - 1)$ -subvariety  $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$  with boundary  $\tilde{S}$  (in the sense of currents) such that the natural projection  $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  restricts to a bijection which is a CR diffeomorphism between  $\tilde{S}$  and  $S$  outside the complex points of  $S$ .*

*Proof.* By Proposition 1, for every  $e_j$ , a continuous function  $\nu'_j$ ,  $C^\infty$  outside  $e_j$ , can be constructed in a neighborhood  $U_j$  of  $e_j$ ,  $j = 1, 2, 3$ , and by Proposition 6, we have an analogous result in a neighborhood of  $\Sigma'$ . Furthermore, from Proposition 15, a smooth function  $\nu''_j$  whose level sets are the leaves of  $\mathcal{F}_j$  can be obtained globally on  $S'_j \setminus \{e_j \cup \Sigma'\}$ . With the functions  $\nu'_j$  and  $\nu''_j$ , and analogous functions near  $\Sigma'$ , then using a partition of unity, we obtain a global smooth function  $\nu_j : S_j \rightarrow \mathbf{R}$  without critical points away from the complex points  $e_j$  and from  $\Sigma'$ .

Let  $\sigma_1$ , resp.  $\sigma_2$  be the two connected, relatively compact components of  $\Sigma \setminus \{h\}$ , according to condition (iv);  $\bar{\sigma}_1$ , resp.  $\bar{\sigma}_2$  are the boundary of  $S_1$ , resp.  $S_2$ , and  $\bar{\sigma}_1 \cup \bar{\sigma}_2$  the boundary of  $S_3$ . We can assume that the three

functions  $\nu_j$  are finite valued and get the same values on  $\overline{\sigma}_1$  and  $\overline{\sigma}_2$ . Hence a function  $\nu : S \rightarrow \mathbf{R}$ .

The submanifold  $S$  being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set  $\tilde{S} = N = \text{gr } \nu = \{(\nu(z), z) : z \in S\}$ . Let  $S_s = \{e_1, e_2, e_3, \overline{\sigma_1 \cup \sigma_2}\}$ .

$\lambda : S \rightarrow \tilde{S}$  ( $z \mapsto \nu((z), z)$ ) is bicontinuous;  $\lambda|_{S \setminus S_s}$  is a diffeomorphism; moreover  $\lambda$  is a CR map. Choose an orientation on  $S$ . Then  $N$  is an (oriented) CR subvariety with the negligible set of singularities  $\tau = \lambda(S_s)$ .

At every point of  $S \setminus S_s$ ,  $d_{x_1} \nu \neq 0$ , then condition (H) (section 3.1.1) is satisfied at every point of  $N \setminus \tau$ .

Then all the assumptions of Theorem 10 being satisfied by  $N = \tilde{S}$ , in a particular case, we conclude that  $N$  is the boundary of a Levi-flat  $(2n-2)$ -variety (with negligible singularities)  $\tilde{M}$  in  $\mathbf{R} \times \mathbb{C}^n$ .

Taking  $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  to be the standard projection, we obtain the conclusion.  $\square$

### 3.5. Generalizations: elementary models and their gluing.

3.5.1. The examples and the proofs of the theorems when  $S$  is homeomorphic to a sphere (sections 3.4) suggest the following definitions.

3.5.2. *Definitions.* Let  $T'$  be a smooth, locally closed (i.e. closed in an open set), connected submanifold of  $\mathbb{C}^n$ ,  $n \geq 3$ . We assume that  $T'$  has the following properties:

(i)  $T'$  is relatively compact, non necessarily compact, and of codimension 2.

(ii)  $T'$  is nonminimal at every CR point.

(iii)  $T'$  does not contain complex manifold of dimension  $(n-2)$ .

(iv)  $T'$  has exactly 2 complex points which are flat and either special elliptic or special 1-hyperbolic.

(v) If  $p \in T'$  is special 1-hyperbolic, the singular orbit  $\Sigma'$  through  $p$  is compact,  $\Sigma' \setminus p$  has two connected components  $\sigma_1, \sigma_2$ , whose closures are homeomorphic to spheres of dimension  $2n-3$ .

(vi) If  $p \in T'$  is special 1-hyperbolic, in the neighborhood of  $p$ , with convenient coordinates, the equation of  $T'$ , up to third order terms is

$$z_n = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \text{Re } z_j^2); \lambda_1 > 1; 0 \leq \lambda_j < 1 \text{ for } j \neq 1$$

or in real coordinates  $x_j, y_j$  with  $z_j = x_j + iy_j$ ,

$$x_n = ((\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2) + \sum_{j=2}^{n-1} ((1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2) + O(|z|^3)$$

(vii) the closures, in  $T'$ ,  $T_1, T_2, T_3$  of the three connected components  $T'_1, T'_2, T'_3$  of  $T' \setminus \Sigma'$  are submanifolds with (singular) boundary. Let  $T''_j$ ,  $j = 1, 2, 3$  be neighborhoods of the  $T'_j$  in  $T'$ .



*up- and down- 1-hyperbolic points.* Let  $\tau$  be the  $(2n - 2)$ -submanifold with (singular) boundary contained into  $T'$  such that either  $\bar{\sigma}_1$  (resp.  $\bar{\sigma}_2$ ) is the boundary of  $\tau$  near  $p$ , or  $\Sigma'$  is the boundary of  $\tau$  near  $p$ . In the first case, we say that  $p$  is *1-up*, (resp. *2-up*), in the second that  $p$  is *down*. If  $T'$  is contained in a small enough neighborhood of  $\Sigma'$  in  $\mathbb{C}^n$ , such a  $T'$  will be called a *local elementary model*, more precisely it defines a *germ of elementary model around  $\Sigma$* .

The union  $T$  of  $T_1$ ,  $T_2$ ,  $T_3$  and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an *elementary model*.  $T$  behaves as a locally closed submanifold still denoted  $T$ .

**3.5.3. Examples of elementary models.** We will say that  $T$  is a *elementary model of type*:

- (a) if it has: two elliptic points;
- (b) if it has: one special elliptic point and one *down*- $\{1\}$ -hyperbolic point;
- (c<sub>1</sub>) if it has: one special elliptic point and one *1-up*- $\{1\}$ -hyperbolic point;
- (c<sub>2</sub>) if it has: one special elliptic point and one *2-up*- $\{1\}$ -hyperbolic point;
- (d<sub>1</sub>) if it has: two special *1-up*- $\{1\}$ -hyperbolic points;
- (d<sub>2</sub>) if it has: two special *2-up*- $\{1\}$ -hyperbolic points;
- (e) if it has: two special *down*- $\{1\}$ -hyperbolic points;

Other configurations are easily imagined.

The prescribed boundary of a Levi-flat hypersurface of  $\mathbb{C}^n$  in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

**3.5.4. Properties of elementary models.** For instance,  $T$  is 1-up and has one special elliptic point, we solve the boundary problem as in  $S_1$  in the proof of Theorem 16.

**Proposition 17.** *Let  $T$  be a local elementary model. Then,  $T$  carries a foliation  $\mathcal{F}$  of class  $C^\infty$  with 1-codimensional CR orbits as compact leaves.*

*Proof.* From the definition at the end of section 3.5.2 and Proposition 6.  $\square$

**3.5.5.**

**Theorem 18.** *Let  $T$  be the elementary model there exists an open neighborhood  $T''$  in  $T'$  carrying a smooth function  $\nu : T'' \rightarrow \mathbb{R}$  whose level sets are the leaves of a smooth foliation.*

*Proof.* By removing small connected open saturated neighborhoods of every special elliptic point, and of  $\Sigma'$ , the singular orbit through every special 1-hyperbolic point  $p$ , we obtain, from  $T \setminus \Sigma'$ , three manifolds  $T_j''$ ,  $j = 1, 2, 3$ , with boundary,

- (a)  $T_1$  and  $T_2$  containing one special elliptic point  $e$  or one special 1-hyperbolic point with the foliations  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , from Propositions 1 and 17,

(b)  $T_3''$  with the foliation  $\mathcal{F}_3$  of codimension 1 given by its CR orbits whose first cohomology group with values in  $\mathbf{R}$  is 0, near  $e$ , or  $p$ . It is easy to show that this later foliation is transversely oriented.

From the Thurston's Stability Theorem (see section 3.4.2),  $T_3''$  is homeomorphic to  $\mathbf{S}^{2n-3} \times [0, 1]$ , foliated as a product, with CR orbits being of the form  $\mathbf{S}^{2n-3} \times \{x\}$  for  $x \in [0, 1]$ ; hence smooth functions  $\nu_1, \nu_2, \nu_3$ , whose level sets are the leaves of the foliations  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  respectively, and using a partition of unity the desired function  $\nu$  on  $T$ .

□

### 3.6.

**Theorem 19.** *Let  $T$  be an elementary model. Then there exists a Levi-flat  $(2n-1)$ -subvariety  $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$  with boundary  $\tilde{T}$  (in the sense of currents) such that the natural projection  $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  restricts to a bijection which is a CR diffeomorphism between  $\tilde{T}$  and  $T$  outside the complex points of  $T$ .*

*Proof.* The submanifold  $T$  being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set  $\tilde{T} = N = \text{gr } \nu = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbf{R} \times \mathbb{C}^{n-1}$ . Let  $T_s$  be the union of the flat complex points of  $T$ .

$\lambda : T \rightarrow \tilde{T}$  ( $z \mapsto \nu((z), z)$ ) is bicontinuous;  $\lambda|_{T \setminus T_s}$  is a diffeomorphism; moreover  $\lambda$  is a CR map. Choose an orientation on  $T$ . Then  $N$  is an (oriented) CR subvariety with the negligible set of singularities  $\tau = \lambda(T_s)$ .

Using Remark 11, at every point of  $T \setminus T_s$ ,  $d_{x_1} \nu \neq 0$ , we see that condition (H) (section 3.1.1) is satisfied at every point of  $N \setminus \tau$ .

Then all the assumptions of Theorem 10 being satisfied by  $N = \tilde{T}$ , in a particular case, we conclude that  $N$  is the boundary of a Levi-flat  $(2n-2)$ -variety (with negligible singularities)  $\tilde{M}$  in  $\mathbf{R} \times \mathbb{C}^n$ .

Taking  $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  to be the standard projection, we obtain the conclusion. □

### 3.7. Gluing of elementary models.

3.7.1. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Remark that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The assumed properties of the submanifold  $S$  in section 2 in  $\mathbb{C}^n$  have a meaning in any complex analytic manifold  $X$  of complex dimension  $n \geq 3$ , and are kept under any holomorphic isomorphism.

We will define a submanifold  $S'$  of  $X$  obtained by gluing of elementary models by induction on the number  $m$  of models. An elementary model  $T$  in  $X$  is the image of an elementary model  $T_0$  in  $\mathbb{C}^n$  by an analytic isomorphism of a neighborhood of  $T_0$  in  $\mathbb{C}^n$  into  $X$ .

3.7.2. Let  $S'$  be a closed smooth real submanifold of  $X$  of dimension  $2n - 2$  which is non minimal at every CR point. Assume that  $S'$  is obtained by gluing of  $m$  elementary models.

a)  $S'$  has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;

b) for every special 1-hyperbolic  $p'$ , there exists a CR-isomorphism  $h$  induced by a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  from a neighborhood of  $p$  in  $T'$  onto a neighborhood of  $p'$  in  $S'$ .

c) for every CR-orbit  $\Sigma_{p'}$  whose closure contains a special 1-hyperbolic point  $p'$ , there exists a CR-isomorphism  $h$  induced by a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  from a neighborhood of  $\Sigma_p = \Sigma'_p \setminus p$  in  $T'$  onto a neighborhood  $V$  of  $\Sigma_{p'}$  in  $S'$ .

Every special 1-hyperbolic point of  $S'$  which belongs to only one elementary model in  $S'$  will be called *free*.

We will define the gluing of one more elementary model to  $S'$ .

3.7.3. *Gluing an elementary model  $T$  of type  $(d_1)$  to a free down-1-hyperbolic point of  $S'$ .* Let  $h_1$  be a CR-isomorphism from a neighborhood  $V_1$  of  $\bar{\sigma}'_1$  induced by a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  onto a neighborhood of  $\sigma_1$  in  $S'$ . Let  $k_1$  be a CR-isomorphism from a neighborhood  $T''_1$  of  $T'_1$  into  $X$  such that  $k_1|_{V_1} = h_1$ .

3.7.4.

**Theorem 20.** *The compact manifold or the manifold with singular boundary  $S'$ , obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of  $X$  in the sense of currents.*

*Proof.* From Theorem 19 and the definition of gluing.  $\square$

3.8. **Examples of gluing.** Denoting the gluing of the two models of type  $(d_1)$  and  $(d_2)$  to a free down-1-hyperbolic point of  $S'$  by:  $\rightarrow (d_1) - (d_2)$ , and the converse by:  $(d_1) - (d_2) \rightarrow$ , and, also, analogous configurations in the same way, we get:

torus:  $(b) \rightarrow (d_1) - (d_2) \rightarrow (b)$ ; the Euler-Poincaré characteristic of a torus is  $\chi(\mathbf{T}^k) = 0$ : 2 special elliptic and 2 special 1-hyperbolic points.

bitorus:  $(b) \rightarrow (d_1) - (d_2) \rightarrow (e) \rightarrow (d_1) - (d_2) \rightarrow (b)$ .

#### 4. CASE OF GRAPHS

(see [DTZ09] for the case of elliptic points only, and dropping the property of the function solution to be Lipschitz).

4.1. We want to add the following hypothesis:  $S$  is embedded into the boundary of a strictly pseudoconvex domain of  $\mathbb{C}^n$ ,  $n \geq 3$ , and more precisely, let  $(z, w)$  be the coordinates in  $\mathbb{C}^{n-1} \times \mathbb{C}$ , with  $z = (z_1, \dots, z_{n-1})$ ,  $w = u + iv = z_n$ , let  $\Omega$  be a strictly pseudoconvex domain of  $\mathbb{C}^{n-1} \times \mathbb{R}_u$  (i.e. the second fundamental form of the boundary  $b\Omega$  of  $\Omega$  is everywhere positive

definite); let  $S$  be the graph  $gr(g)$  of a smooth function  $g : b\Omega \rightarrow \mathbb{R}_v$ . notice that  $b\Omega \times \mathbb{R}_v$  contains  $S$  and is strictly pseudoconvex.

Assume that  $S$  is a *horned sphere* (section 3.4), *satisfying the hypotheses of Theorem 16*. Denote by  $p_j$ ,  $j = i, \dots, 4$  the complex points of  $S$ . Our aim is to prove

4.2.

**Theorem 21.** *Let  $S$  be the graph of a smooth function  $g : b\Omega \rightarrow \mathbb{R}_v$ . Let  $Q = (q_1, \dots, q_4) \in b\Omega$  be the projections of the complex points  $P = (p_1, \dots, p_4)$  of  $S$ , respectively. Then, there exists a continuous function  $f : \overline{\Omega} \rightarrow \mathbb{R}_v$  which is smooth on  $\overline{\Omega} \setminus Q$  and such that  $f|_{b\Omega} = g$ , and  $M_0 = \text{graph}(f) \setminus S$  is a smooth Levi flat hypersurface of  $\mathbb{C}^n$ . Moreover, each complex leaf of  $M_0$  is the graph of a holomorphic function  $\phi : \Omega' \rightarrow \mathbb{C}$  where  $\Omega' \subset \mathbb{C}^{n-1}$  is a domain with smooth boundary (that depends on the leaf) and  $\phi$  is smooth on  $\overline{\Omega}'$ .*

The natural candidate to be the graph  $M$  of  $f$  is  $\pi(\tilde{M})$  where  $\tilde{M}$  and  $\pi$  are as in Theorem 16. We prove that this is the case proceeding in several steps.

#### 4.3. Behaviour near $S$ .

4.3.1. Assume that  $D$  is a strictly pseudoconvex domain and that  $S \subset bD$ .

Recall ([HL75][Theorem 10.4]: Let  $D$  be a strictly pseudoconvex domain of  $\mathbb{C}^n$ ,  $n \geq 3$  with boundary  $bD$ ,  $\Sigma \subset bD$  be a compact connected maximally complex smooth  $(2d-1)$ -submanifold with  $d \geq 2$ . Then,  $\Sigma$  is the boundary of a uniquely determined relatively compact subset  $V \subset \overline{D}$  such that  $\overline{V} \setminus \Sigma$  is a complex analytic subset of  $D$  with finitely many singularities of pure dimension  $\leq d-1$ , and near  $\Sigma$ ,  $\overline{V}$  is a  $d$ -dimensional complex manifold with boundary.

$V$  is said to be the solution of the boundary problem for  $\Sigma$ .

4.3.2.

**Lemma 22** ([DTZ09]). *Let  $\Sigma_1, \Sigma_2$  be compact connected maximally complex  $(2d-1)$ -submanifolds of  $bD$ . Let  $V_1, V_2$  be the corresponding solutions of the boundary problem. If  $d \geq 2$ ,  $2d \geq n+1$  and  $\Sigma_1 \cap \Sigma_2 = \emptyset$ , then  $V_1 \cap V_2 = \emptyset$ .*

Let  $\Sigma$  be a CR orbit of the foliation of  $S \setminus P$ . Then  $\Sigma$  is a compact maximally complex  $(2n-3)$ -dimensional real submanifold of  $\mathbb{C}^n$  contained in  $bD$ . Let  $V = V_\Sigma$  be the solution of the boundary problem corresponding to  $\Sigma$ . From Theorem 16,  $V = \pi(\tilde{V})$ , where  $\tilde{V} = (\tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{x\})$  for suitable  $x \in (0, 1)$ , the projection on the  $x$ -axis being finite, we can always assume that it lies into  $(0, 1)$ . Moreover  $\pi|_{\tilde{V}}$  is a biholomorphism  $\tilde{V} \cong V$  and  $M \setminus S \subset D$ .

Let  $\Sigma_1, \Sigma_2$  be two distinct orbits of the foliation of  $S \setminus P$ , and  $\overline{V}_1, \overline{V}_2$  the corresponding leaves, then, from Lemma 22,  $\overline{V}_1 \cap \overline{V}_2 = \emptyset$ .

4.3.3. Assume that  $S$  satisfies the full hypotheses of Theorem 21.

Set  $m_1 = \min_S g$ ,  $m_2 = \max_S g$  and  $r \gg 0$  such that

$$D = \Omega \times [m_1, m_2] \subset \subset \mathbf{B}(\mathbf{r}) \cap (\Omega \times i\mathbb{R}_v)$$

where  $\mathbf{B}(\mathbf{r})$  is the ball  $\{|(z, w)| < r\}$ .

4.3.4.

**Lemma 23.** *Let  $p \in S$  be a CR point. Then, near  $p$ ,  $M$  is the graph of a function  $\phi$  on a domain  $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_u$  which is smooth up to the boundary of  $U$ .*

*Proof.* Near  $p$ , each CR orbit  $\Sigma$  is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and  $V_\Sigma$  as the graph of the local holomorphic extension of this function. From Hopf lemma,  $V$  is transversal to the strictly pseudoconvex hypersurface  $d\Omega \times i\mathbb{R}_v$  near  $p$ . Hence the family of the  $V_\Sigma$ , near  $p$ , forms a smooth real hypersurface with boundary on  $S$  that is the graph of a smooth function  $\phi$  from a relative open neighborhood  $U$  of  $p$  on  $\overline{\Omega}$  into  $\mathbb{R}_v$ . Finally, Lemma 22 guarantees that this family does not intersect any other leaf  $V$  from  $M$ .  $\square$

4.3.5.

**Corollary 24.** *If  $p \in S$  is a CR point, each complex leaf  $V$  of  $M$ , near  $p$ , is the graph of a holomorphic function on a domain  $\Omega_V \subset \mathbb{C}_z^{n-1}$ , which is smooth up to the boundary of  $\Omega_V$ .*

#### 4.4. Solution as a graph of a continuous function.

4.4.1. Recall results of Shcherbina [Shc93] from:

(a) the Main Theorem:

Let  $G$  be a bounded strictly convex domain in  $\mathbb{C}_z \times \mathbb{R}_u$  ( $z \in \mathbb{C}$ ) and  $\varphi : bG \rightarrow \mathbb{R}_v$  be a continuous function. Then the following properties hold, where  $\Gamma = gr$ , and  $\hat{\Gamma}(\varphi)$  means polynomial hull of  $\Gamma(\varphi)$ :

(a<sub>i</sub>) the set  $\hat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$  is the union of a disjoint family of complex discs  $\{D_\alpha\}$ ;

(a<sub>ii</sub>) for each  $\alpha$ , there is a simply connected domain  $\Omega_\alpha \subset \mathbb{C}_z$  and a holomorphic function  $w = f_\alpha$ , defined on  $\Omega_\alpha$ , such that  $D_\alpha$  is the graph of  $f_\alpha$ .

(a<sub>iii</sub>) For each  $f_\alpha$ , there exists an extension  $f_\alpha^* \in C(\overline{\Omega}_\alpha)$  and  $bD_\alpha = \{(z, w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f_\alpha^*(z)\}$ .

(b)

**Lemma 25.** *Let  $\{G_n\}_{n=0}^\infty$ ,  $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$ , be a sequence of bounded strictly convex domains such that  $G_n \rightarrow G_0$ . Let  $\{\varphi_n\}_{n=0}^\infty$ ,  $\varphi_n : \partial G_n \rightarrow \mathbb{R}_v$  be a sequence of continuous functions such that  $\Gamma(\varphi_n) \rightarrow \Gamma(\varphi_0)$  in the Hausdorff metric. Then, if  $\Phi_n$  is the continuous function :  $\overline{G}_n \rightarrow \mathbb{R}_v$  such that  $\hat{\Gamma}(\varphi) = \Gamma(\Phi)$ , we have  $\Gamma(\Phi_n) \rightarrow \Gamma(\Phi_0)$  in the Hausdorff metric.*

(c)

**Lemma 26.** *Let  $\mathcal{U}$  be a smooth connected surface which is properly embedded into some convex domain  $G \subset \mathbb{C}_z \times \mathbb{R}_u$ . Suppose that near each point of this surface, it can be defined locally by the equation  $u = u(z)$ . Then the surface  $\mathcal{U}$  can be represented globally as a graph of some function  $u = U(z)$ , defined on some domain  $\Omega \subset \mathbb{C}_z$ .*

4.4.2.

**Proposition 27.**  *$M$  is the graph of a continuous function  $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ .*

*Proof.* We will intersect the graph  $S$  with a convenient affine subspace of real dimension 4 to go back to the situation of Shcherbina.

Fix  $a \in (\mathbb{C}_z^{n-1} \setminus 0)$  and, for a given point  $(\zeta, \xi) \in \Omega$ , with  $\zeta \in \mathbb{C}_z^{n-1}$  and  $\xi \in \mathbb{R}_u$ , let  $H_{(\zeta, \xi)} \subset \mathbb{C}_z^{n-1} \times \{\xi\}$  be the complex line through  $(\zeta, \xi)$  in the direction  $(a, 0)$ . Set:

$$L_{(\zeta, \xi)} = H_{(\zeta, \xi)} + \mathbb{R}_u(0, 1), \quad \Omega_{(\zeta, \xi)} = L_{(\zeta, \xi)} \cap \Omega, \quad S_{(\zeta, \xi)} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap S$$

Then  $S_{(\zeta, \xi)}$  is contained in the strictly convex cylinder

$$(H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap (b\Omega \times i\mathbb{R}_v)$$

and is the graph of  $g|_{b\Omega_{(\zeta, \xi)}}$ .

From  $(a_{ii})$ , the polynomial hull of  $S_{(\zeta, \xi)}$  is a continuous graph over  $\overline{\Omega}_{(\zeta, \xi)}$ . Consider  $M = \pi(\tilde{M})$  and set

$$M_{\zeta, \xi} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap M.$$

It follows that  $M_{\zeta, \xi}$  is contained in the polynomial hull  $\hat{S}_{(\zeta, \xi)}$ . From  $(a_{iii})$ ,  $\hat{S}_{(\zeta, \xi)}$  is a graph over  $\overline{\Omega}_{(\zeta, \xi)}$  foliated by analytic discs, so  $M_{\zeta, \xi}$  is a graph over a subset  $U$  of  $\overline{\Omega}_{(\zeta, \xi)}$ .

Every analytic disc  $\Delta$  of  $\hat{S}_{(\zeta, \xi)}$  had its boundary on  $S_{(\zeta, \xi)}$ . Since all the complex points of  $S$  are isolated,  $b\Delta$  contains a CR point  $p$  of  $S$ ; from Lemma 23, near  $p$ ,  $M_{\zeta, \xi}$  is a graph over  $\overline{\Omega}_{(\zeta, \xi)}$ . Near  $p$ ,  $\Delta$  is contained in  $M_{\zeta, \xi}$ , then in a closed complex analytic leaf  $V_\Sigma$  of  $M$ ; so  $\Delta \subset V_\Sigma \subset M$ ; but  $\Delta \subset H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)$ ; then:  $\Delta \subset M_{\zeta, \xi}$ . Consequently, near  $p$ ,  $M_{\zeta, \xi} = \hat{S}_{(\zeta, \xi)}$ .

It follows that  $M$  is the graph of a function  $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ .

One proves, using (b), that  $f$  is continuous on  $\Omega$ , whence on  $\overline{\Omega} \setminus Q$ , by Lemma 23. Then continuity at every  $q_j$  is proved using the *Kontinuitätsatz* on the domain of holomorphy  $\Omega \times i\mathbb{R}_v$ .  $\square$

**4.5. Regularity.** The property:  $M \setminus P = (p_1, \dots, p_4)$  is a smooth manifold with boundary results from:

4.5.1.

**Lemma 28.** *Let  $U$  be a domain of  $\mathbb{C}_z^{n-i} \times \mathbb{R}_u$ ,  $n \geq 2$ ,  $f : U \rightarrow \mathbb{R}_v$  a continuous function. Let  $A \subset \text{graph}(f)$  be a germ of complex analytic set of codimension 1. Then  $A$  is a germ of complex manifold which is a graph of over  $\mathbb{C}_z^{n-i}$ .*

*Proof.* Assume that  $A$  is a germ at 0. Let  $h \in \mathcal{O}_{n+1}$ ,  $h \neq 0$  such that  $A = \{h = 0\}$ . For  $\varepsilon \ll 1$ , let  $\mathbf{D}_\varepsilon$  be the disc  $\{z = 0\} \cap \{|w| < \varepsilon\}$ , then  $A \cap \mathbf{D}_\varepsilon = \{0\}$ , i.e.  $A$  is  $w$ -regular.

Let  $\pi : \mathbb{C}_{z,w}^n \rightarrow \mathbb{C}_z^{n-1}$  be the projection. The local structure theorem for analytic sets gives:

*for some neighborhood  $U$  of 0 in  $\mathbb{C}_z^{n-1}$ , there exists an analytic hypersurface  $\Delta \subset U$  such that:  $A_\Delta = A \cap ((U \setminus \Delta) \times \mathbf{D}_\varepsilon)$  is a manifold;*

*$\pi/A_\Delta \rightarrow U \setminus \Delta$  is a  $d(\in \mathbb{N})$ -sheeted covering.*

It is easy to show that the covering  $\pi : A_\Delta \rightarrow U \setminus \Delta$  is trivial.

Then we may define  $d$  holomorphic functions  $\tau_1, \dots, \tau_d : U \setminus \Delta \rightarrow \mathbb{C}$  such that  $A_\Delta$  is the union of the graphs of the  $\tau_j$ . By the Riemann extension theorem, the functions  $\tau_j$  extend as holomorphic functions  $\tau_j \in \mathcal{O}(U)$ . Suppose that  $\tau_j \neq \tau_k$ , for  $j \neq k$ , then for some disc  $\mathbf{D} \subset U$  centered at 0, we have  $\tau_j|_{\mathbf{D}} \neq \tau_k|_{\mathbf{D}}$ , then  $(\tau_j - \tau_k)|_{\mathbf{D}}$  vanishes only at 0. But, from the hypothesis, in restriction to  $\mathbf{D}$ ,  $\{Re(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|_{\mathbf{D}} = \{0\}$ , impossible.  $\square$

4.6.

*Proof of the Theorem 21.* Consider the foliation of  $S \setminus P$  given by the level sets of the smooth function  $\nu : S \rightarrow [0, 1]$  (sections 2.3 and 2.7) and set  $L_t = \{\nu = t\}$  for  $t \in (0, 1)$ . Let  $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$  be the complex leaf of  $M$  bounded by  $L_t$ .

By Proposition 27,  $M$  is the graph of a continuous function over  $\Omega$ , and, by Lemma 28, each leaf  $V_t$  is a complex smooth hypersurface and  $\pi|_{V_t}$  is a submersion.

Since  $\Omega$  is strictly convex, as in Shcherbina (see 4.4.1, c)),  $\pi|_{V_t}$  is 1-1, then, by Corollary 24,  $\pi$  sends  $V_t$  onto a domain  $\Omega_t \subset \mathbb{C}_z^{n-1}$  with smooth boundary. Let

$$\pi_u : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_u$$

$$\pi_v : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_v$$

then  $\pi_u|_{L_t} = a_t \cdot \pi|_{L_t}$  and  $\pi_v|_{L_t} = b_t \cdot \pi|_{L_t}$  where  $a_t, b_t$  are smooth functions on  $b\Omega_t$ . Moreover  $b\Omega_t$ ,  $a_t, b_t$  depend smoothly on  $t$ .

If  $(z_t, w_t) \in M$ , then  $w_t$  varies on  $V_t$ , so  $w_t$  is the holomorphic extension of  $a_t + ib_t$  to  $\Omega_t$ . In particular  $u_t$  and  $v_t$  are smooth in  $(z, t)$ , from the Bochner-Martinelli formula.

$\frac{\partial u_t}{\partial t}$  is harmonic on  $\Omega_t$  for each  $t$  and has a smooth extension on  $b\Omega_t$ .

From Lemma 23 and Corollary 24,  $\frac{\partial u_t}{\partial t}$  does not vanish on  $b\Omega_t$ . Since the CR orbits  $L_t$  are connected from Proposition 14,  $b\Omega_t$  is also connected,

hence  $\frac{\partial u_t}{\partial t}$  has constant sign on  $b\Omega_t$ . Then, by the maximum principle, also  $\frac{\partial u_t}{\partial t}$  on  $\Omega_t$  and, in particular does not vanish. This implies that  $M \setminus S$  is the graph of a smooth function over  $\Omega$  which smoothly extends to  $\overline{\Omega} \setminus Q$ .

From Proposition 27,  $M$  is the graph of a continuous function over  $\overline{\Omega}$ .  $\square$

#### 4.7. Elementary smooth models.

**4.7.1. Definition.** An elementary smooth model in  $\mathbb{C}^n$  is an elementary model in the sense of section 3.5.2 and satisfying the further condition which makes sense from Theorem 21:

(G) Let  $(z, w)$  be the coordinates in  $\mathbb{C}^{n-1} \times \mathbb{C}$ , with  $z = (z_1, \dots, z_{n-1})$ ,  $w = u + iv = z_n$ , let  $\Omega$  be a strictly pseudoconvex domain of  $\mathbb{C}^{n-1} \times \mathbb{R}_u$ ; assume that  $T'$  is the graph of a smooth function  $g : b\Omega \rightarrow \mathbb{R}_v$ .

#### 4.7.2.

**Theorem 29.** Let  $T$  be an elementary smooth model. Then, there exists a continuous function  $f : \overline{\Omega} \rightarrow \mathbb{R}_v$  which is smooth on  $\overline{\Omega} \setminus Q$  and such that  $f|_{b\Omega} = g$ , and  $M_0 = \text{graph}(f) \setminus S$  is a smooth Levi flat hypersurface of  $\mathbb{C}^n$ ; in particular,  $S$  is the boundary of the hypersurface  $M = \text{graph}(f)$

*Proof.* similar to the proof of Theorem 21.  $\square$

**4.7.3. Gluing of elementary smooth models.** In an open set of  $\mathbb{C}^n$ , a coordinate system  $(z, w)$  of  $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$  defines an  $(n-1, 1)$ -frame.

To define the gluing of elementary models (section 3.7) we considered a CR-isomorphism from an open set of  $\mathbb{C}^n$  induced by a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  onto an open set of  $\mathbb{C}^n$ . To define the gluing of elementary smooth models, we have to consider a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  onto an open set of  $\mathbb{C}^n$  sending an  $(n-1, 1)$ -frame of  $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$  onto an  $(n-1, 1)$ -frame of  $\mathbb{C}_{z'}^{n-1} \times \mathbb{R}_{u'}$ .

As in section 3.7.1, we will define a submanifold  $S'$  of  $X$  obtained by gluing of elementary smooth models by induction on the number  $m$  of models. An elementary smooth model  $T$  in  $X$  is the image of an elementary smooth model  $T_0$  of  $\mathbb{C}^n$  by an analytic isomorphism of a neighborhood of  $T_0$  in  $\mathbb{C}^n$  into  $X$ .

*Gluing an elementary smooth model  $T$  of type  $(d_1)$  to a free down-1-hyperbolic point of  $S'$ .*

Every elementary smooth model is contained in a cylinder  $b\Omega \times \mathbb{R}_v$  determined by  $\Omega$  and an  $(n-1, 1)$ -frame. Two sets  $\Omega$  are *compatible* if either they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-isomorphism  $h_1$  from a neighborhood  $V_1$  of  $\overline{\sigma}'_1$  induced by a holomorphic isomorphism of the ambient space  $\mathbb{C}^n$  onto a neighborhood of  $\sigma_1$  in  $S'$ . Let  $k_1$  be a CR-isomorphism from a neighborhood  $T'''_1$  of  $T'_1$  into  $X$  such that  $k_1|_{V_1} = h_1$ , and there exists a common  $(n-1, 1)$ -frame on which the



corresponding sets  $\Omega$  are compatible. The existence of such a situation is possible as the example of the horned (almost everywhere) smooth sphere shows (Theorem 21).

Remark that the gluing implies that the obtained submanifold  $S'$  is  $C^0$  and smooth except at the complex points.

Other gluings are obtained in a similar way. Hence:

**Theorem 30.** *The manifold  $S'$  obtained by gluing of elementary smooth models is of class  $C^0$ , and smooth except at the complex points.*

**Corollary 31.** *The manifold  $S'$  is the boundary of a manifold  $M$  of class  $C^0$  whose interior is a Levi-flat smooth hypersurface.*

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